Efficient Calculation of Absolute Orientation with Outlier Rejection

Esfandiar Bandari[***], Norman Goldstein[**], Issa Nesnas[*], Max Bajracharya[*]
[***]RIACS, NASA Ames Research Center
MacDonald Dettwiler and Associates
[*]Jet Propulsion Laboratory

Abstract

This paper addresses a high level spatio-temporal problem, namely "absolute orientation", which arises in visual-odometry (using stereo), or registering two models created by different Structure from Motion (SFM) reconstructions. We compare the very popular method due to Horn[1] using quaternions and our own independently derived method using the orthogonal rotation matrix **R**. We also introduce a *novel* approach for outlier rejection using spectral clustering[2, 3].

Introduction: Absolute orientation, Horn[1], is a higher level spatio-temporal problem that arises in visual-odometry, graphics, mesh registration, and extending 3D models from SFM or stereo. The problem statement is simple: given two sets of *matched* points, **p** and **q** $\in \mathbb{R}^3$, find the *similarity* transform that takes **p** into **q**, rejecting all outliers. By similarity, we mean the translation, rotation and scale such that $\mathbf{q}_i = s * \mathbf{R} * \mathbf{p}_i + \mathbf{t}^i$, here s is the scale ambiguity (arising if **p** and **q** are SFM outputs), **R** is the 3x3 rotation matrix, and **t** is the translation.

Several authors have already addressed this problem. Perhaps the absolute reference on absolute orientation is Horn[1], where a unit quaternion (on SO^3) represents the rotation. Horn, Hilden and Negahdaripour[4] too address the absolute orientation problem, but use the rotation matrix **R**. This approach is later rejected for reasons such as:

- Rotation matrices have too many parameters – i.e. 9 instead of 4 parameters for quaternions.
- Rotation matrices have "strange" constraints, i.e. $\mathbf{R} * \mathbf{R}^T$ has to be the 3x3 identity matrix, and det(**R**) = 1.
- Lastly, quaternions are ideal for interpolative purposes compared to the rotation matrix **R**.[5]

The first two objections are rather mute. First, to solve for proper quaternion, Horn creates a 9x9 matrix and solves for variables bilinear in the elements of **Q**. Second, a unit quaternion **Q** has to satisfy the constraint $\mathbf{Q} \bullet \mathbf{Q}^* = 1$ where **Q*** is the conjugate of **Q**, and $\bullet$ represents quaternion multiplication; Horn addresses this using a Lagrange multiplier. Regarding interpolation, Michaels and Boult are quite correct. While rotation matrices do not lend themselves, to interpolation, transforming them to Rodrigez representation is trivial.

In the next section we provide a fast and robust LSQ technique to address the absolute orientation problem. The subsequent section describes a new approach to outlier rejection, based on spectral clustering[2], which has proven extremely robust. In practice, of course, the outlier rejection is done first, followed by the LSQ solution.

LSQ Solution. The problem is to minimize the objective function

$$G(\mathbf{R}, \mathbf{t}, s) = \Sigma_i |\mathbf{p}_i - s * \mathbf{R} * \mathbf{q}_i - \mathbf{t}|^2 \qquad (1)$$

with respect to $s \in \mathbb{R}$, $\mathbf{t} \in \mathbb{R}^3$ and **R** a rotation matrix. Using a Lagrange Multiplier formulation, the stationary points are where the gradient of $W(\mathbf{R}, \mathbf{t}, s) = \mathbf{G}(\mathbf{R}, \mathbf{t}, s) - \frac{1}{2}\,\text{trace}\,(\mathbf{M}\,(\mathbf{I} - \mathbf{R}^T * \mathbf{R})$ is 0. Here, **R** is now *any* 3x3 matrix, and **M** is a symmetric 3x3 matrix of Lagrange multipliers. Umeyama[6] extends the problem by adding in another Lagrange multiplier for the det(**R**)=1. As we will see shortly, our approach provides a simpler solution to the problem.[ii] The **t** component of the gradient shows that:

$$\mathbf{t} = \mathbf{m}(\mathbf{p}) - s * \mathbf{R} * \mathbf{m}(\mathbf{q}) \qquad (1.1)$$

where **m(p)** and **m(q)** are the centroids of the **p** and **q** sets of points. Substituting this back into Eq 1, we get the same equation except $\mathbf{t} = \mathbf{m}(\mathbf{p}) = \mathbf{m}(\mathbf{q}) = \mathbf{0}$, but the new $\mathbf{q'}_i$ and $\mathbf{p'}_i$ have the centroid **m**(**p**) and **m**(**q**) subtracted from $\mathbf{q}_i$ and $\mathbf{p}_i$ respectively. The gradient of the new W with respect to s and **R** gives the two equations:

$$s \Sigma_i |\mathbf{q'}_i|^2 = \text{trace}\,(\mathbf{R}\,\mathbf{K}) \qquad (2.1)$$

$$\mathbf{K} = \mathbf{M}\,\mathbf{R}^T / s \qquad (2.2)$$

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[i] Notation: **UPPERCASE BOLD** $\in \mathbb{R}^{3x3}$, **lowercase bold** $\in \mathbb{R}^3$, lowercase letter $\in \mathbb{R}$.

[ii] We received the reference to Umeyama's work after our derivation. Although outliers are not addressed in this work, the mathematical treatment is very elegant, and interested readers are encouraged to examine this work.

where $\mathbf{K} = \Sigma_i \mathbf{q'}_i \mathbf{p'}_i^T$. Using $\mathbf{R}\,\mathbf{R}^T = \mathbf{I}$ and **M** being symmetric and equation 2.2 (or the polar decomposition of **K**) we get $\mathbf{M}/s = \text{sqrt}(\mathbf{K}\,\mathbf{K}^T)$ where the square root has the eigenvalues $(\sigma_1 > \sigma_2 > \varepsilon\sigma_3)$. Hence, the σ's are also the singular values of **K**, and ε=sign(det(K)). The use of ε ensures that **R** is a rotation matrix and not a reflection. Using singular value decomposition of $\mathbf{K}=\mathbf{U}\mathbf{S}\mathbf{V}^T$ then $\mathbf{M}=\mathbf{U}\mathbf{S'}\mathbf{U}^T$, where $\mathbf{S'}=\text{diag}(\sigma_1, \sigma_2, \varepsilon\sigma_3)$ and substituting these into equation 2.2 and we have:

$$\mathbf{R} = \mathbf{W}\,\mathbf{U}^T \qquad (3.1)$$

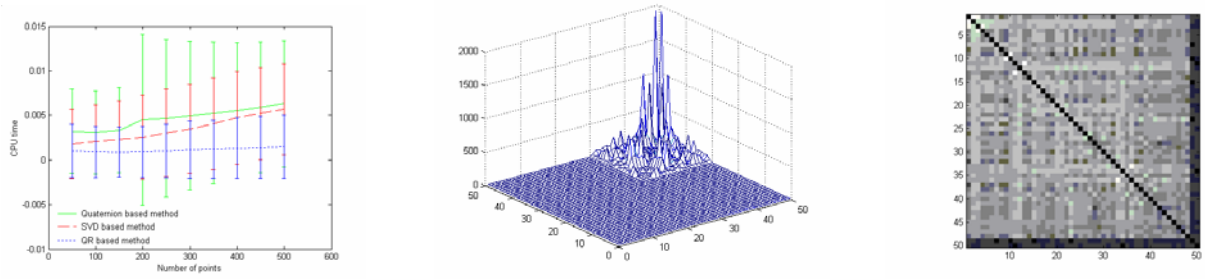
where **W=V** when $\varepsilon = 1$, and **W** is equal to **V** with a sign reversal on each element of the 3rd column when $\varepsilon = -1$. Moreover, using the **R** and **K** into 2.1 we have

$$s = (\sigma_1 + \sigma_2 + \varepsilon\,\sigma_3) / \Sigma_i\, |\mathbf{q'}_i|^2 \qquad (3.2)$$

Equations 3.1, 3.2, and 1.1 (used in order) provide the least square solution for the similarity transform sought.

Outlier Rejection: Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be the symmetric matrices that contain the distance between every two points in **p'** and in **q'** respectively. If **W** is the element by element division of $\mathbf{D}_1$ by $\mathbf{D}_2$ except for the diagonals which remain 0, then the elements of **W** are the scale factor, s, plus added noise. The exception being the rows and columns of **W** corresponding to outliers (i.e., mismatched points). One popular method of rejecting the outliers is RANSAC[7]. But using spectral clustering, it can be shown that the eigenvector of **W**, $\mathbf{e}_1$, corresponding to **W**'s largest Eigen value, is almost blockwise constant. That is, for all elements of $\mathbf{e}_1$ only those that correspond to the rows (or columns) of **W** containing the outliers have values different from the rest. The remaining values of $\mathbf{e}_1$ are almost all constant (with minor differences due to noise). This gives us the following simple process to reject the outliers: 1) construct **W**. 2) find the eigenvector $\mathbf{e}_1$ for **W**'s largest Eigen value. 3) sort $\mathbf{e}_1$, keeping track of the changes in indices. 4) create the permutation matrix **Q** by rearranging the columns of the identity matrix, **I**, according the changes in $\mathbf{e}_1$ indices. 5) create the matrix $\mathbf{X} = \mathbf{Q}^T\,\mathbf{W}\,\mathbf{Q}$. **X** is diagonally block structured and s can easily be identified. Note that one can also use RANSAC on the block diagonal elements of **X**, thus increasing its speed by reducing the number of outliers.

Experimental Results: Figure 1a shows the results of our Monte-Carlo simulations. Three techniques, Horn, our SVD based derivation and QR decomposition of the matrix R derived using least squares were compared. While the QR method was the fastest –see figure 1.a – it also generated the most unreliable solutions. Our approach was faster than Horn's technique (or its equivalents), with the sum square error of the two methods being indistinguishable. Figures 1.b and 1.c show the structure of the **X** matrix for 40% outliers and (its element wise log) for 96% outliers!



(a) **(b)** **(c)**

Figure 1. (a) The results of Monte Carlo simulation for Horn, QR, and our SVD based method. (b) the block diagonal structure of the **X** matrix after spectral clustering for 40% outliers and (c) element wise log of matrix **X** for 96% outliers.

References:

[1] Horn, P. K. B., "Closed-Form Solution of Absolute Orientation Using Unit Quaternions," JOSA A, V4, N4, 1987.

[2] Brand, M. nd Huang K., "A Unifying Theorem for Spectral Embedding and Clustering", Mistubishi Electric Research Laboratory Tech Report TR-2002-42, Nov. 2002

[3] Ng, A., Jordan, M. and Weiss, Y., "On spectral clustering: Analysis and an algorithm", NIPS 2001.

[4] Horn, B., Hilden, H., and Negahdaripour, S. "Closed-Form Solution of Absolute Orientation Using Orthonormal Matrices," JOSA A, Vol. 5, No. 7, 1988.

[5] Micheals, R. and Boult, T. "A New Closed Form Approach to the Absolute Orientation Problem", VAST Lab, Lehigh University, Bethlehem, PA, 1999

[6] Umeyama, S. et al. "Least-squares estimation of transformation parameters between two point patterns" IEEE PAMI, April 1991. vol.13, no.4, p. 376-80

[7] Fischler M. and Bolles, R.: 1981, "RANSAC: *random sample consensus: A paradigm for model fitting with applications to image analysis and automated cartography*" Communications of the ACM 26, 381-395.